Characterizing the Gröbner Bases of Generic Ideals

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Introduction

Algebraic geometry is the study of finding zeros of multivariate polynomials. Computation allows for the calculation of Gröbner bases from a set of polynomials. Moreno-Socías conjectured that the initial ideal generated by generic polynomials relative to graded reverse lexicographic ordering is weakly reverse lexicographic. This has been proven up to $n = 3$, as well as for certain special cases. Numerous other mathematicians have used the conjecture to show that it implies various other conjectures, including Fröberg's conjecture.

Objectives

The complexity of the subject divided our work into two parts: understanding the background information, and manipulating the concepts and ideals. Working with a conjecture made previously by fellow students, our goals included:

- Testing the validity of the conjecture
- Reformulating the conjecture if needed
- Confirming the presence of the conjecture in the 2-variable case
- Proving that the conjecture implies Moreno-Socías' conjecture
- Proving that the conjecture implies Fröberg's conjecture on Hilbert Series
- Explicitly characterizing the Gröbner bases of "random" generic generators
- Proving the conjecture in the 3-variable case as well as the n-variable case

Methods

We first studied the basic concepts of computational algebraic geometry and the proofs of Moreno-Socías' conjecture in the 2-variable case.

We then worked with a computer program to look at the Gröbner bases generated by three generic polynomials of varying degrees in three variables in order to understand what was going on in the 3-variable case.

Key Definitions

Let $R = K[x_1, \dots, x_n]$ be the polynomial ring in n variables over the infinite field K

- A **generic polynomial** $f = \sum_m \alpha_m m$ where m runs over all monomials of degree d with $\alpha_m \epsilon K$ satisfies
  - If $m \neq m'$ then $\alpha_m \neq \alpha_{m'}$.
  - The set $\{\alpha_m\}$ is algebraically independent over K.
- A **generic ideal** I is generated by generic polynomials $f_1, \dots, f_r$, where $f_i = \sum_{m_{f_i}} \alpha_{m_{f_i}} m_{f_i}$ such that
  - $\alpha_{m_{f_i}} \neq \alpha_{m_{f_j}}$ for $i \neq j$.
  - The set $\left\{\alpha_{m_{f_i}} \middle| 1 \leq i \leq r\right\}$ is algebraically independent over K.
- **Grevlex Order**: we say that $x^\alpha > x^\beta$ if $\deg(\alpha) > \deg(\beta)$ or they are of the same degree and the rightmost nonzero entry of $\alpha - \beta$ is negative.
- A monomial ideal $J \subset$ R is **weakly reverse lexicographic** if whenever $x^\alpha \in J$ is a minimal generator, then every monomial of the same degree preceding x^α in grevlex order is also in J.

Conjecture

CCOZ Conjecture

Let I be generated by three generic polynomials of degrees $d_1 \leq d_2 \leq d_3$ with $J = in(I)$ the initial ideal. Let $X \subseteq$ J be the set of minimal generators, with X_s the minimal generators of degree s. Then

1. If $x^\alpha, x^\beta \in X_s$ and $x^\alpha > x^\gamma > x^\beta$ in grevlex order then $x^\gamma \in J_s$.
2. Max X_{s+1} is determined by multiplying the first element in the complement of J_s by the last nonzero element in the multi-degree of that element.
3. There is a reduced Gröbner basis G of I such that for each $f \in G$, a monomial m of f follows the leading term of f, in grevlex ordering, if and only if m has the same total degree of f and $m \notin$ J.

Results

- The CCOZ conjecture implies the Moreno-Socías conjecture in 3 variables and can be extended to the n-variable case.
- If we know the initial ideal, we can construct the Gröbner basis.
- The bases depend on the starting degrees of the generic polynomials.

Future Research

In general, this research allows us to better characterize Gröbner bases of generic ideals, but we are still left with several open-ended questions that could be investigated in more depth in the future:

- Can we determine the end point of each X_s so that we can more explicitly describe the bases?
- Can we prove the CCOZ conjecture is true?
- Can the CCOZ conjecture be used to prove Fröberg's conjecture on Hilbert series without Moreno-Socías' conjecture?

References

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